ACCUMULATION AND REMOVAL OF GROUNDWATER RIDGES

S. N. Muminov

Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, Vol. 8, No. 3, pp. 134-137, 1967

Solutions are known [1, 2] for the motion of groundwater during and after influx along a strip, for soils of any depth. The case of infinitely deep soil has also been considered [3] for a rectangular region. Here results are given for a rectangular region of finite depth.

The linearized equation for the groundwater motion is

$$\frac{\partial h}{\partial t} = a\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right) + f(x, y, t)$$

$$\left(a = \frac{kh'}{\sigma},\ f(x, y, t) = \frac{W(x, y, t)}{\sigma}\right), \tag{1}$$

where k is the filtration coefficient, h' is the mean height of the soil flow, σ is the saturation deficit, and W is the difference between infiltration and evaporation. Consider an unbounded region in the xy plane; over a certain part there is influx. The initial form for the surface of the groundwater flow is $h(x, y, 0) = h_0(x, y)$, and the solution to (1) is

$$h(x, y, t) = \frac{1}{4\pi at}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\frac{-r^2}{4at} h_0(x_1, y_1)\,dx_1 dy_1 +$$

$$+ \frac{1}{\pi a4}\int_0^t \frac{dt_1}{t-t_1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\frac{-r^2}{4a(t-t_1)} f(x_1, y_1, t_1)\,dx_1 dy_1,$$

$$r^2 = (x - x_1)^2 + (y - y_1)^2. \tag{2}$$

Decay of a rise. If infiltration and evaporation are neglected, the first term in (2) defines the shape. Consider the case of a rectangular parallelepiped:

$$h_0(x, y) = H_0 = \text{const}, \quad |x| \leqslant R, \quad |y| \leqslant R_1,$$

$$h_0(x, y) = H_1 = \text{const}, \quad |x| > R_1, \quad |y| > R_1.$$

Then the solution is

$$h(x, y, t) = H_1 + \frac{H_0 - H_1}{4}\left[\operatorname{erf}\frac{R-x}{2\sqrt{at}} + \operatorname{erf}\frac{R+x}{2\sqrt{at}}\right]\times$$

$$\times\left[\operatorname{erf}\frac{R-y}{2\sqrt{at}} + \operatorname{erf}\frac{R+y}{2\sqrt{at}}\right]. \tag{3}$$

We introduce the dimensionless quantities

$$\frac{x}{R} = \xi, \quad \frac{y}{R_1} = \eta, \quad \tau = \frac{4at}{R^2}, \quad n = \frac{R_1}{R},$$

$$U(\xi, \eta, \tau) = \frac{h(x, y, t) - H_1}{H_0 - H_1},$$

whereupon (3) becomes

$$U(\xi, \eta, \tau) = \frac{1}{4}\left[\operatorname{erf}\frac{1-\xi}{\sqrt{\tau}} + \operatorname{erf}\frac{1+\xi}{\sqrt{\tau}}\right]\times$$

$$\times\left[\operatorname{erf}\frac{(1-\eta)n}{\sqrt{\tau}} + \operatorname{erf}\frac{(1+\eta)n}{\sqrt{\tau}}\right]. \tag{4}$$

For $n = \infty$ (case of a strip $-R \leq x \leq R$) we have

$$U(\xi, \tau) = \frac{1}{2}\left[\operatorname{erf}\frac{1-\xi}{\sqrt{\tau}} + \operatorname{erf}\frac{1+\xi}{\sqrt{\tau}}\right]. \tag{5}$$

Formula (4) becomes as follows for the center of the rectangle for $\xi = 0$, $\eta = 0$:

$$U(\tau) = \operatorname{erf}\frac{1}{\sqrt{\tau}}\operatorname{erf}\frac{n}{\sqrt{\tau}}. \tag{6}$$

If n = 1 we have a square, while for n = 2, 3, ... we have rectangles, and for $n = \infty$ we have a strip. Also, $U(\tau) \approx 4n/\pi\tau$ for τ large.

Figure 1 shows curves for the decline in the maximum of $U = U(\tau;\ n)$ derived from (6) via tables for n = 1, 2, 3, 4, and ∞. The curves fall steeply initially, but then more slowly. Even for n = 2, the curve at the start is quite close to that for $n = \infty$, while at τ of 7-8 the curve for n = 2 lies almost halfway between those for n = 1 and $n = \infty$, though later it tends to approach the line for n = 1.

The line for n = 3 lies very close to those for n = 4 and $n = \infty$ at τ of 3-4, but some separation occurs for τ of 6-7. The line for n = 4 lies even closer to that for $n = \infty$ at τ of 5-6 but deviates appreciably at τ of 7-8. The lines for n of 3 and 4 are close to the line for $n = \infty$, as the table shows.

τ	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = \infty$
0	1	1	1	1	0
0.5	0.910	0.954	0.954	0.954	0.954
1	0.709	0.834	0.842	0.842	0.842
1.5	0.554	0.735	0.750	0.751	0.751
2.0	0.466	0.651	0.681	0.683	0.683
2.5	0.394	0.581	0.624	0.628	0.628
3.0	0.342	0.525	0.578	0.584	0.585
3.5	0.302	0.478	0.537	0.548	0.550
4.0	0.270	0.438	0.502	0.517	0.520
5.0	0.224	0.375	0.446	0.467	0.473
6.0	0.190	0.327	0.399	0.427	0.436
7.0	0.165	0.291	0.362	0.393	0.407
8.0	0.146	0.260	0.331	0.364	0.382
9.0	0.131	0.237	0.305	0.340	0.362

Figure 2 gives curves for a rise that is initially represented by a square column, n = 1 in (4), for several η, and also for $n = \infty$ [from (5)] for τ of 1 and 4. The curves for n = 1 show that U initially falls rapidly as τ increases for $\xi = 0$ and that there is an appreciable rise in the groundwater for $\xi > 1$, e.g., for $\xi = 2$. The same general picture applies for n > 1, but with a less rapid decline.

The contours for n = 1 tend to become circles as we recede from the center, and they become even closer to circles for τ large.

In large irrigated areas, the rises due to irrigation may not have time to vanish not only from one flooding to the next but even from year to year, e.g., when there are several large floodings in a season or much atmospheric precipitation.

Consider the case R = 100 m, $a = 2000\ \text{m}^2/\text{day}$; then $t = 1.25\ \tau$. We deduce the time needed for the height of the peak to fall by a factor of 10, i.e., to give U = 0.1 (Fig. 2), as $\tau = 11.1$ for a square and $\tau = 123$ for $n = \infty$, i.e., 14 and 154 days respectively.

A further instance where the solution to (1) is obtained in simple form is when

$$h_0(x, y) = A\exp\{-\alpha^2 x^2 - \beta^2 y^2\}, \tag{7}$$

we have

$$h(x, y, t) = \frac{A}{\sqrt{\theta(\alpha, t)\,\vartheta(\beta, t)}}\exp\left(\frac{-\alpha^2 x^2}{\theta(\alpha, t)} + \frac{-\beta^2 y^2}{\vartheta(\beta, t)}\right),$$

$$\theta(\alpha, t) = 1 + 4\alpha^2 at,$$

$$\vartheta(\beta, t) = 1 + 4\beta^2 at. \tag{8}$$

If $h_0(x, y)$ is the sum of several components, h(x, y, t) will be the sum of the corresponding components. The peak in the rise decays, but the other ordinates may increase somewhat, since the total volume of the incompressible fluid is constant. One rise acts on another to increase the ordinates and delay the decline.

Irrigation of a rectangular area. The irrigation is assumed to occur at a constant rate, i.e., the $f(x, y, t)$ of (2) is independent of t. Then

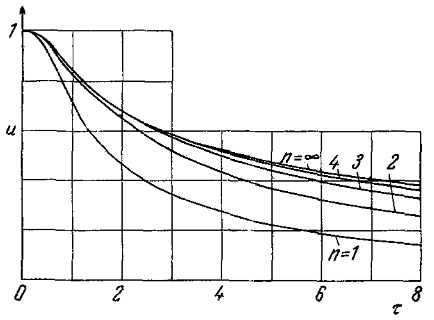

Fig. 1

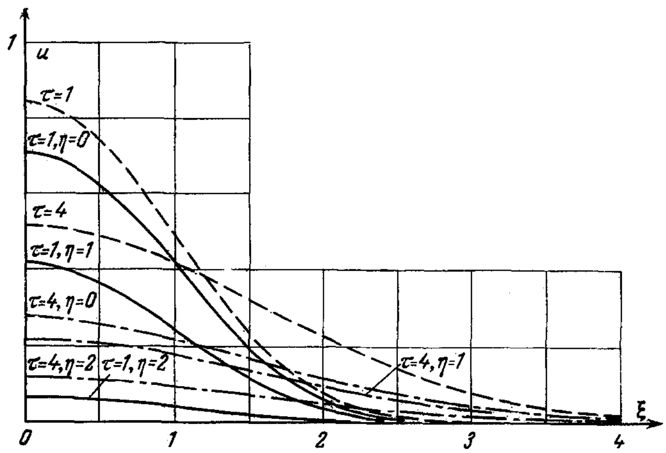

Fig. 2

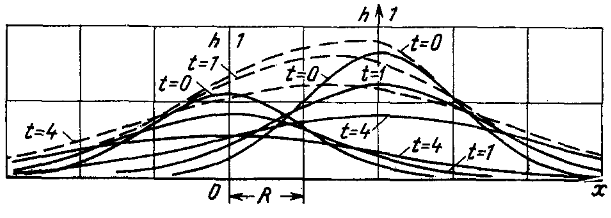

Fig. 3

we can put $t - t_1 = t'$ in the integral and, taking $h_0(x, y) = 0$, we have

$$h(x, y, t) = \frac{1}{4\pi a}\int_0^t \frac{dt'}{t'}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\frac{-[(x-\xi)^2+(y-\eta)^2]}{4at'}\times$$

$$\times f(\xi, \eta)\, d\xi\, d\eta. \tag{9}$$

We assume that the irrigation is performed from the surface of a rectangle $-R \le x \le R$, $-R_1 \le y \le R_1$ above which $W = \varepsilon =$ constant, with $W = 0$ outside the rectangle. Then (9), with $h_0(x, y) = H_0$, gives

$$h(x, y, t) = \frac{\varepsilon}{4a\pi\sigma}\int_0^t \frac{dt'}{t'}\int_{-R}^{R} \exp\frac{-(x-\xi)^2}{4at'}\, d\xi \times$$

$$\times \int_{-R_1}^{R_1} \exp\frac{-(y-\eta)^2}{4at'}\, d\eta + H_0.$$

By analogy with the decay problem, we have

$$h(x, y, t) = H_0 + \frac{\varepsilon}{4\sigma}\int_0^t \left(\operatorname{erf}\frac{R-x}{2\sqrt{at}} + \operatorname{erf}\frac{R+x}{2\sqrt{at}}\right)\times$$

$$\times\left(\operatorname{erf}\frac{R_1-y}{2\sqrt{at}} + \operatorname{erf}\frac{R_1+y}{2\sqrt{at}}\right) dt. \tag{10}$$

At the center of the rectangle ($x = y = 0$)

$$h(0, 0, t) = H_0 + \frac{\varepsilon}{\sigma}\int_0^t \operatorname{erf}\frac{R}{2\sqrt{at}}\operatorname{erf}\frac{R_1}{2\sqrt{at}}\, dt.$$

The arguments of the cofactors in the integral are large for t small, while these cofactors themselves are close to 1 and may be put as

$$\operatorname{erf} x = 1 - \operatorname{erfc} x, \qquad \operatorname{erfc} x = \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-\xi^2}\, d\xi.$$

We apply the asymptotic formula

$$\operatorname{erfc} x = \frac{e^{-x^2}}{x\sqrt{\pi}}\left(1 - \frac{1}{2x^2} + \cdots\right),$$

in which the error does not exceed the first term discarded. We discard $x^2/2$ to get

$$h(0, 0, t) - H_0(t) \approx$$

$$\approx \varepsilon\left\{\frac{1}{\sigma}t - \sqrt{\frac{t}{\pi}}\,[f(\alpha) + f(\alpha_1)] - 2tf_1(\beta)\right\},$$

$$f(\alpha) = \frac{e^{-\alpha^2}}{3\alpha^2} + \frac{2e^{-\alpha^2}}{3} + \frac{2\sqrt{\pi}\alpha}{3}\operatorname{erfc}\alpha,$$

$$f_1(\beta) = \frac{e^{-\beta}}{\beta} + e^{-\beta} - \beta E_i(-\beta),$$

$$\alpha = \frac{R}{2\sqrt{at}}, \qquad \alpha_1 = \frac{R_1}{2\sqrt{at}}, \qquad \beta = \frac{R^2 + R_1^2}{4at}.$$

This shows that the center rises uniformly at a rate ε/σ for t small. The duration of this stage of uniform accumulation is dependent on the size of the irrigated area, being largest for a given R when $R_1 = \infty$ (strip). In that case, the exact equation has [1] the simple form

$$U(0, \tau) = \tau - \left[\left(\tau + \frac{1}{2}\right)\operatorname{erf}\frac{1}{2\sqrt{\tau}} - \sqrt{\frac{\tau}{\pi}}\exp\frac{-1}{4\tau}\right],$$

$$\tau = \frac{at}{R^2}.$$

The line $u(0, \tau)$ deviates from a straight line for $\tau > 0.1$.

Examples. In two adjacent parts of width 2R we have simultaneous irrigation at rates ε and 1.5ε. The water is cut off at a certain time. Here we represent the rise at that instant via (7), while the decay is defined by (8). Figure 3 shows the curves for t of 0, 1, and 4. The dashed lines represent the results of addition for the curves for the free surface. It is clear that the water table rises at large distances from the irrigated areas.

I am indebted to P. Ya. Kochina for assistance and direction in this work.

REFERENCES

1. P. Ya. Polubarinova-Kochina, Theory of Groundwater Motion [in Russian], Gostekhizdat, 1952.

2. N. N. Kochina, "A two-dimensional problem in the decay of a groundwater rise in a layer of infinite depth," PMM, vol. 15, no. 6, 1951.

3. N. N. Kochina, "Some aspects of the spatial flow of groundwater," PMM, vol. 17, no. 3, 1953.

8 July 1966 Novosibirsk